# Designing Circular and Non-Circular Gears for FDM 3d-Printing

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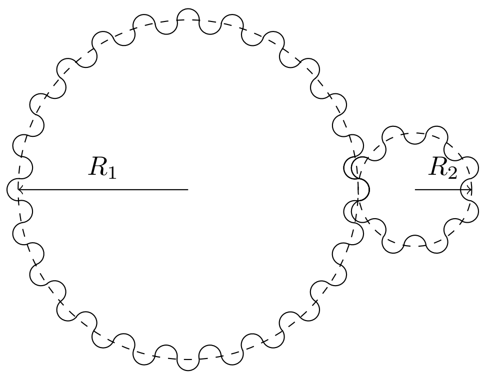


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# Contents

# 1. Tooth Profile

Since FDM 3d printed gears are weak and therefore torque and transmission efficiency is not as important we are going to use a circular *tooth profile*. Please note that this profile is not optimal but it makes it incredibly easy to calculate and draw non circular gears. It is sufficient for most 3d printed parts. If this is not suitable for your design take a look at involute gears.

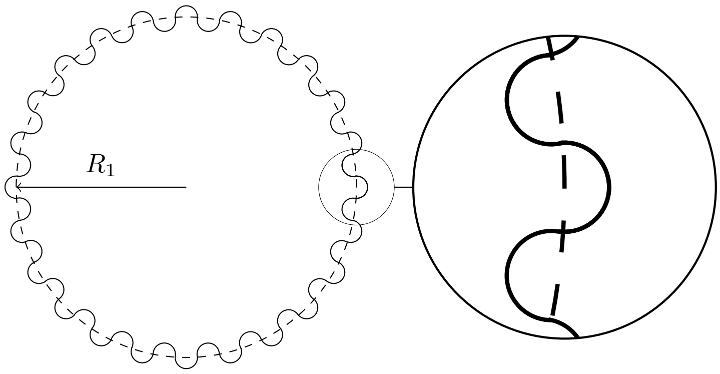



Figure 1: Gear profile construction using circles

## 1.1. Drawing a Gear

Start by drawing the outline of the gear. As we will see in section 3.3 this can be any convex shape. As an example we will take a look at a circular gear. Then, using a compass, mark points with distance $r$ on the outline of your shape and draw a circle with radius $r$ on every second mark. We will see in section 3 and in the examples how to calculate $r$.

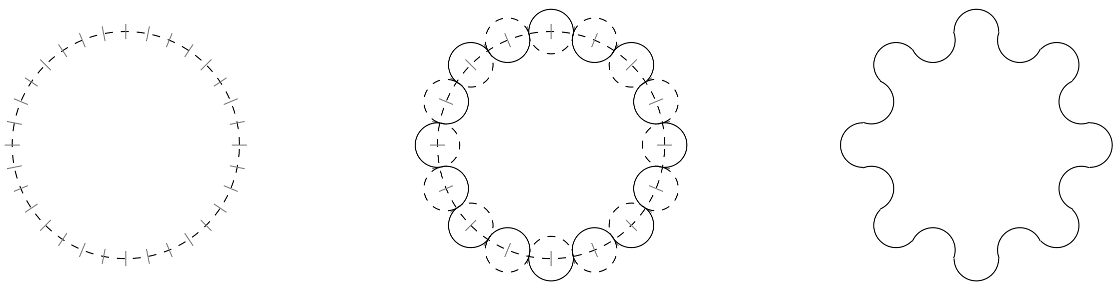


Figure 2: Drawing a circular gear

# 2. Notation

To analyze different gear shapes and calculate the *tooth size* $r$ we need to grasp what a gear is mathematically. As seen in the previous chapter, a gear is constituted by the *gear shape* together with the number of teeth, the size (radius) of each tooth, a set containing the centerpoints of each tooth and the *centerpoint* of the gear itself. This leads to the following definition:

**Definition** 2.1
Let $\gamma : [0, 2\pi] \mapsto \mathbb{R}^2$ be a simple, closed and rectifiable curve such that a convex domain $\Omega$ with $\partial\Omega = \gamma[[0, 2\pi]]$ exists.
We'll call a tuple $(\gamma, n, m_0, M)$ where $n \in \mathbb{N}$ a *gear* if it is possible to find a polygon $\Gamma$ with $4n$ vertices $(m_i)_{i\in\mathbb{Z}_{4n}} \in \gamma[[0, 2\pi]]$ such that $\exists!\, r \in \mathbb{R} \forall i \in \mathbb{Z}_{4n} : d(m_i, m_{i+1}) = r \wedge r > 0$.

At this point it is not clear if a gear is well-defined as Definition 2.1 demands only a single tooth-centerpoint $m_0$ and not a set containing every tooth-centerpoint. The fact that $(m_i)_{i\in\mathbb{Z}_{4n}}$ can be represented by $m_0$ is one of the findings of Theorem 3.3.
If $\gamma$ is a circle we will sometimes replace it with its radius $R \in \mathbb{R}$. Furthermore $M$ is defined as the center point of $\Omega$. $M := \frac{1}{\lambda^2(\Omega)} \int_\Omega x d\lambda^2$. Since $M$ is defined by $\gamma$ it is not required. However we will only omit $M$ if it is 0.

**Definition** 2.2
Two gears $A$ and $B$ *fit together* if

$$r_A = r_B.$$

**Definition** 2.3
Two gears $A$ and $B$ are of the *same size* or shape, if it is possible to map the image of $\gamma_A$ to the image of $\gamma_B$ without changing its size:

$$\exists\iota(\forall x, y \in \mathbb{R}^2 : d(\iota(x), \iota(y)) = d(x, y)) \wedge (\iota[\gamma_A[[0, 2\pi)]] = \gamma_B[[0, 2\pi)]]). \tag{2.1}$$

We call two gears $A$ and $B$ *similar* and write $A \sim B$ if they are of the same size and

$$n_A = n_B$$
$$r_A = r_B$$
$$\iota(m_{A,0}) = m_{B,0}$$

where $\iota$ is an isometric function that satisfies (2.1).
Furthermore $A = B$ if $A \sim B$ and $\iota = \mathrm{id}$.

**Proof**

For this definition to make sense $\sim$ needs to be reflexive, symmetric and transitive.

"$A \sim A$":

Let $\iota$ be the identity function.

"$A \sim B \implies B \sim A$":

As $\iota$ is an isometric function it is injective and therefore its left inverse $\iota^{-1}$ exists. Using $\iota^{-1}$, $B \sim A$.

"$A \sim B \land B \sim C \implies A \sim C$":

The composition of two isometric functions is an isometric function and therefore $A \sim C$. □

# 3. Calculating Gear and teeth Size

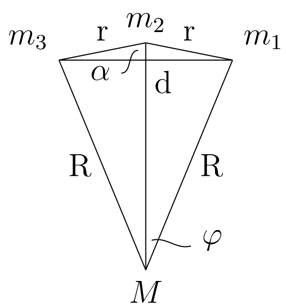
## 3.1. Circular Gears

We have already seen how to draw a circular gear $K = (R, n, m_0, M)$ in section 1.1. However $r$ has to be chosen in a way that it is possible to find a set of $4n$ points $(m_i)_{i\in\mathbb{Z}_{4n}} \in \gamma[[0, 2\pi)] : d(m_i, m_{i+1}) = r \wedge r > 0$.

**Theorem** 3.1
Given a circle $K$ with radius $R$, $(R, n \in \mathbb{N}^*, m_0 \in K, M)$ is a gear. Let $\varphi = \frac{\pi}{n}$ then

$$r = \frac{R \cdot \sin(\varphi)}{\cos(\frac{\varphi}{2})}. \tag{3.1}$$

**Proof**
Let $K$ be a circle with radius $R$ and center $(0, 0)$ and $n \in \mathbb{N}^*$ we can, using polar coordinates, construct a sequence $(m_i)_{i\in\mathbb{Z}_{4n}}$, $m_i = \left(R, i \cdot \frac{\pi}{2n}\text{rad}\right)$ of $4n$ points on $K$. They are evenly spaced and $r := d(m_1, m_2)$.




Let $\alpha := \sphericalangle m_3 m_2 m_1$, $\varphi := \sphericalangle m_1 M m_3$ and $d := \overline{m_3 m_1}$. Using the inscribed angle theorem(Weisstein 2020), we know that $\alpha = \frac{2\pi - \varphi}{2}$. By drawing a line from $M$ to $m_2$ we get two right triangles. $\overline{Mm_2}$ also bisects $\varphi$, $\alpha$ and $d$. Thus

$$\frac{d}{2} = r \cdot \sin\left(\frac{\alpha}{2}\right) \tag{3.2}$$

and

$$R = \frac{d}{2\sin\left(\frac{\varphi}{2}\right)}. \tag{3.3}$$

By substituting in $\alpha$ in eq. (3.2) and using $\cos(x) = \sin(\frac{\pi}{2} - x)$, we see that

$$R = \frac{r \cdot \cos\left(\frac{\varphi}{4}\right)}{\sin(\frac{\varphi}{2})}. \tag{3.4}$$

□

## 3.2. Rectangular Gears

**Theorem** 3.2
Let $K$ be a rectangle with side lengths $a$ and $b$, center point M and corners $A, B, C$ and $D$. Let

$$r := \frac{a+b}{2n}, \tag{3.5}$$

$(K, n, A, M)$ is a gear if $4r \mid a \wedge 4r \mid b$.

**Proof** It is easy to see that Theorem 3.2 holds true. □

## 3.3. Other Non Circular Gears

**Theorem** 3.3
Let $\gamma$ be a simple, closed curve, $\gamma : [0, 2\pi] \to \mathbb{R}^2$.
$(\gamma, n, \gamma(0), M)$ is a gear. Furthermore $r$ and $(\varphi_i)_{i \in \mathbb{Z}_{4n}}$ where $\gamma(\sum_{j=0}^{i} \varphi_j) = m_i$ can be calculated using a system of $4n$ equations:

$$\begin{aligned} \varphi_0 &= 0 \\ \forall i \in \{0, \dots 4n-2\} : \quad d_2\left(\gamma\left(\sum_{j=0}^{i} \varphi_j\right), \gamma\left(\sum_{j=0}^{i+1} \varphi_j\right)\right) &= r \\ d_2\left(\gamma\left(\sum_{j=0}^{4n-1} \varphi_j\right), \gamma\left(\varphi_0\right)\right) &= r. \end{aligned} \tag{3.6}$$

using the following constraints:

$$\forall i \in \{0, \dots 4n\} : \quad \varphi_i \in [0, 2\pi]$$
$$\sum_{i=0}^{4n-1} \varphi_i < 2\pi$$

**Proof**

We first note that as $\gamma$ and $d_2$ are continuous, using the intermediate value theorem,

$$r \leq \sup_{t\in[0,2\pi]} \{d_2(\gamma(t), \gamma(0))\} \implies \exists\varphi : d_2(\gamma(\varphi), \gamma(0)) = r.$$

We now construct $r_0 = 0$, and ,with a bit more work, $r_1$ such that constructing $(\varphi_i)_{i\in\mathbb{Z}_{4n}}$ points using

$$\varphi_0 = 0$$
$$\forall i \in \{0, \dots 4n-2\} : \quad d_2\left(\gamma\left(\sum_{j=0}^{i}\varphi_j\right), \gamma\left(\sum_{j=0}^{i+1}\varphi_j\right)\right) = r_1$$

results in $d_\gamma(0, \sum_{j=0}^{4n-1}\varphi_i) > L(\gamma)$.
Where

$$d_\gamma : [0,\infty)\times[0,\infty) \to \mathbb{R}_0^+, \quad (a,b) \mapsto \left\lfloor \frac{2\pi}{d_2(a,b)} \right\rfloor L(\gamma) + \int_{\min(a,b) \mod 2\pi}^{\max(a,b) \mod 2\pi} |\gamma'|dt$$

and

$$L(\gamma) \coloneqq \int_0^{2\pi} |\gamma'|dt.$$

Thus, using the intermediate value theorem once more, it is possible to find $r$ such that the resulting $(\varphi_i)_{i\in\mathbb{Z}_{4n}}$ satisfy eq. (3.6).
Furthermore, as $d_\gamma$ is increasing, $r$ is unique. □

## 3.4. Examples

**Example 3.4.1** Circular gears with given gear ratio and centerdistance
Lets say we want a 3:1 gear reduction and the center points of each gear should be 30mm apart. We can now start to build a system of equations:

$$\begin{aligned} R_1 + R_2 &= 30 \\ n_1 &= 3n_2 \end{aligned}$$

Where $R_1$ and $R_2$ are the radii of the two gears and $n_1$ and $n_2$ denote the number of teeth on each. Using eq. (3.1) we also know that:

$$\begin{aligned} \varphi_1 &= \frac{2\pi}{n_1} \\ R_1 &= \frac{r * \cos\left(\frac{\varphi_1}{4}\right)}{\sin(\frac{\varphi_1}{2})} \\ \varphi_2 &= \frac{2\pi}{n_2} \\ R_2 &= \frac{r * \cos\left(\frac{\varphi_2}{4}\right)}{\sin(\frac{\varphi_2}{2})} \end{aligned}$$

If we specify the number of teeth on either one of them lets say

$$n_2 = 16.$$

we can solve the system of equations using wolframalpha, maple or another computerprogram to find a unique solution. In this example we get:

$$\begin{aligned} r &= 1.471832758 \\ R_1 &= 22.49196236 \\ R_2 &= 7.508037642 \\ n_1 &= 48 \\ n_2 &= 16 \\ \varphi_1 &= 0.1308996939 \\ \varphi_2 &= 0.3926990818 \end{aligned}$$

If $r$ is too small/large change the number of teeth you specified.

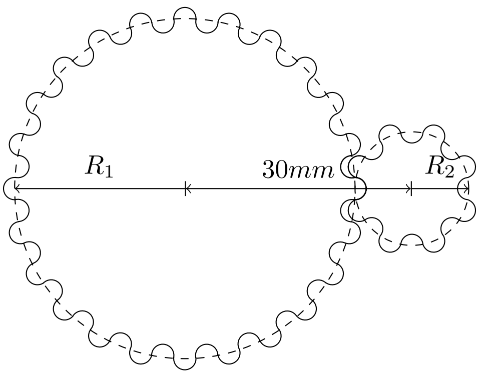



Figure 3: Resulting Gears

**Example 3.4.2** Internal ring gear

A internal ring gear $K_r$ with outer diameter $D_r = 5$ should be calculated such that a smaller gear $K$ can be used to archive a 5 : 1 gear reduction. The two shafts should be 15$mm$ apart.
As the two shafts are 15$mm$ apart we know that

$$R_r - R = 15.$$

Furthermore

$$n_r = 5 \cdot n$$

to achieve the required 5 : 1 ratio. Using eq. (3.1) we also know that:

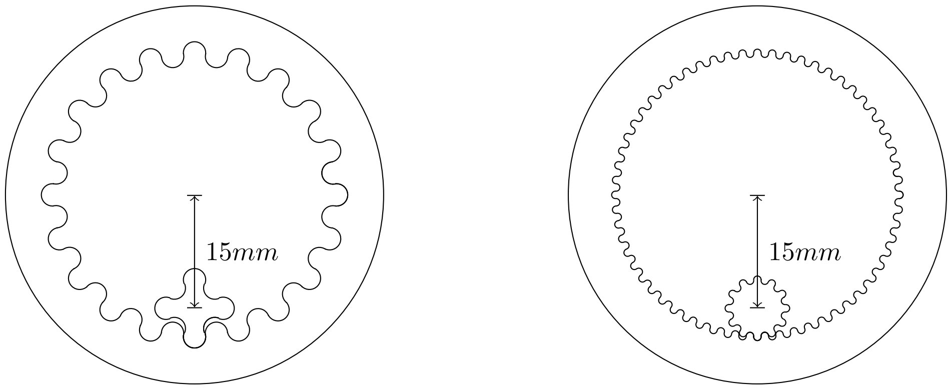
$$\begin{aligned}
\varphi_r &= \frac{2\pi}{n_r} \\
R_r &= \frac{r * \cos\left(\frac{\varphi_r}{4}\right)}{\sin(\frac{\varphi_r}{2})} \\
\varphi &= \frac{2\pi}{n} \\
R &= \frac{r * \cos\left(\frac{\varphi}{4}\right)}{\sin(\frac{\varphi}{2})}
\end{aligned}$$

If we specify the number of teeth on either one of them lets say

$$n = 8,$$

we can solve the system of equations using wolframalpha, maple or another computerprogram to find a unique solution. In this example we get:

$$\begin{aligned} r &= 1.474527091 \\ R_r &= 18.77908826 \\ R &= 3.779088256 \\ n_r &= 40 \\ n &= 8 \\ \varphi_r &= 0.1570796327 \\ \varphi &= 0.7853981635 \end{aligned}$$




(a) $n = 8$ (b) $n = 24$

Figure 4: Resulting internal ring gears for different $n$

**Example 3.4.3** Planetary gearbox (epicyclic gear train)
A planetary gearbox with not rotating ring gear and a gear ratio of 5:1 should be calculated. Let $S$ be the sun gear, $P$ the planet gear and $R$ the outer ring gear and $C$ the carrier.
We know that for planetary gear systems

$$n_R = 2 \cdot n_P + n_S \implies \frac{n_S}{n_P}\omega_S + (2 + \frac{n_S}{n_P})\omega_R - 2(1 + \frac{n_S}{n_P})\omega_C = 0$$

where $\omega$ denotes the angular momentum. As the the outer ring gear $R$ is not rotating $\omega_R = 0$. We can deduce that

$$\begin{aligned} \frac{n_S}{n_P}\omega_S &= 2(1 + \frac{n_S}{n_P})\omega_C \\ \frac{\omega_S}{\omega_C} &= \frac{2(1 + \frac{n_S}{n_P})}{\frac{n_S}{n_P}}. \end{aligned}$$

As $S$ will be used as the input and $C$ as the output we know that

$$\frac{\omega_S}{\omega_C} = \frac{5}{1}$$

and because the gears need to touch each other the radii have to satisfy

$$R_R = 2 \cdot R_P + R_S.$$

We will use eq. (3.1) to get one more equation per gear.[1] After specifying the radius of the outer ring gear $R_R = 30$ and $n_S = 24$ we can solve the system of equation to get a unique solution. Since we want 4 planet gears, the sun gear and the outer ring gear have to be symmetrical in both the $x$ and $y$ direction. This is guaranteed if $4 \mid n_S$.

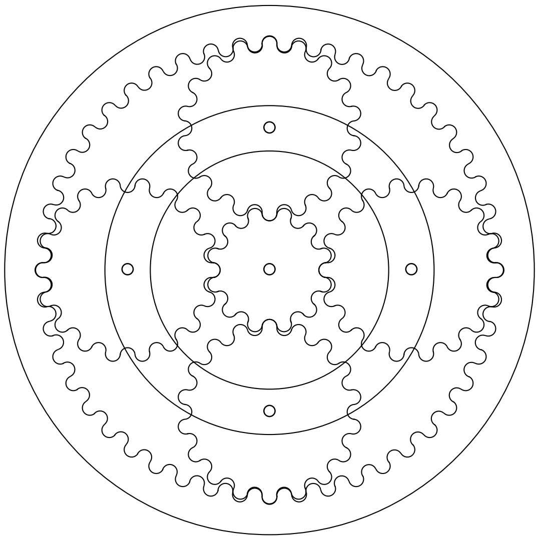


Figure 5: Resulting planetary gear system

**Example 3.4.4** Rectangular/square gear with given side length
A square gear with side length $s = 22mm$ should be calculated.
We choose $r = 22/12$ according to Theorem 3.2 such that $4r \mid s$. Now, as

$$r = \frac{2s}{2n}$$

we can calculate the number of teeth required as

$$n = \frac{s}{r}.$$

[1]Take a look at example 3.4.1 to see this step in more detail.

Thus $n = 12$. Note that each corner of the square/rectangle is a center-point of one of the $2n$ circles required to construct the gear.
Depending on whether we want the corners to be teeth or slots in between two teeth, two different gear outlines are possible. The possible gears for this example can be seen in fig. 6. Figure 7 shows the resulting gears when using a smaller teeth-size $r = 22/24$.

(a) Corners are not teeth (b) Corners are teeth

Figure 6: Resulting rectangular gear for $r = 22/12$

(a) Corners are not teeth (b) Corners are teeth

Figure 7: Resulting rectangular gear for $r = 22/24$

**Example 3.4.5** Rectangular and circular gears with given gear ratio and center-distance

A square gear $S$ and a circular gear $C$ that fit together should be calculated given a center-distance $d = 15mm$.
We start building a system of equation:
For $S$ and $C$ to fit together we know that $r_S = r_C$. W therefore use $r = r_S = r_C$ below. Given the center-distance $d = 15mm$ leads to

$$\frac{s}{2} + R_C = 15mm.$$

Using Theorem 3.1 with $\varphi = \pi/(2n_C)$, we get

$$R_C = \frac{r \cdot \cos\left(\frac{\varphi}{2}\right)}{\sin(\varphi)}.$$

Furthermore, using Theorem 3.2 we know that

$$r = \frac{s}{n_S}.$$

As both $n_S$ and $n_C$ have to be integers we use the symmetry of $C$ to deduce that

$$n_S = 2kn_C \quad k \in \mathbb{N}.$$

We can choose $k$ in a way that the resulting gear and tooth size satisfies our needs. For this example let $k = 1$. Therefore $n_S = 2n_C$.
For $C$ to be symmetric we use

$$n_C = 12$$

as our final equation. Solving this system of equations (numerically) results in:

$$\begin{aligned} r &= 0.7635571812 \\ R &= 5.837313828 \\ s &= 18.32537235 \\ n_C &= 12 \\ n_S &= 24. \end{aligned}$$

The resulting gears can be seen in fig. 8 below.

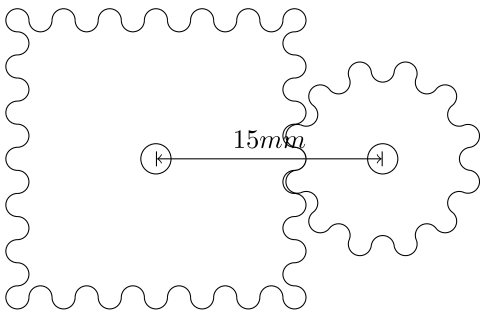



Figure 8: Resulting rectangular gear with fitting circular gear (scale 1:2)

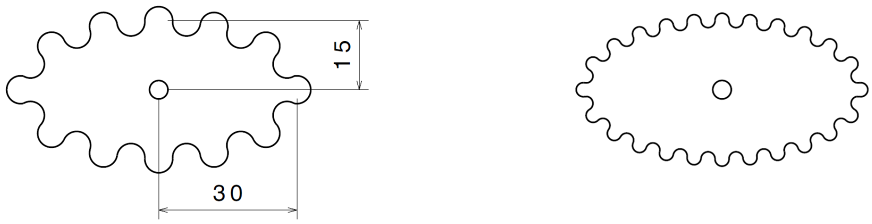
**Example 3.4.6** Elliptical gear with given size and number of teeth
A gear in the shape of an ellipse $(x/a)^2 + (y/b)^2 = 1$ with $a = 30mm$ and $b = 15mm$ and $n = 12$ teeth should be constructed.
We start with a parametrization

$$\gamma : [0, 2\pi) \to \mathbb{R}^2, \quad t \mapsto \begin{pmatrix} a\cos(t) \\ b\sin(t) \end{pmatrix}$$

of the given ellipse. As $\gamma$ satisfies the requirements of Theorem 3.3, eq. (3.6) can be used to calculate the tooth-size $r$ and the position of each tooth. As the system of equations required consists of $4n = 48$ equations we can not list them here in the same way we did in previous examples. The maple code used to solve this system of equations can be found in appendix A.1.
Solving this system leads to:

$$\begin{aligned}
&r = 3.024112686, && && \\
&\varphi_1 = 0., && \varphi_2 = .1990159115, && \varphi_3 = .1809691691, \\
&\varphi_4 = .1599940895, && \varphi_5 = .1430803899, && \varphi_6 = .1305321589, \\
&\varphi_7 = .1213147333, && \varphi_8 = .1145167936, && \varphi_9 = .1095068588, \\
&\varphi_{10} = .1058685767, && \varphi_{11} = .1033313491, && \varphi_{12} = .1017233388, \\
&\varphi_{13} = .1009429577, && \varphi_{14} = .1009429577, && \varphi_{15} = .1017233388, \\
&\varphi_{16} = .1033313491, && \varphi_{17} = .1058685767, && \varphi_{18} = .1095068588, \\
&\varphi_{19} = .1145167936, && \varphi_{20} = .1213147333, && \varphi_{21} = .1305321589, \\
&\varphi_{22} = .1430803899, && \varphi_{23} = .1599940895, && \varphi_{24} = .1809691691, \\
&\varphi_{25} = .1990159115, && \varphi_{26} = .1990159115, && \varphi_{27} = .1809691691, \\
&\varphi_{28} = .1599940895, && \varphi_{29} = .1430803899, && \varphi_{30} = .1305321589, \\
&\varphi_{31} = .1213147333, && \varphi_{32} = .1145167936, && \varphi_{33} = .1095068588, \\
&\varphi_{34} = .1058685767, && \varphi_{35} = .1033313491, && \varphi_{36} = .1017233388, \\
&\varphi_{37} = .1009429577, && \varphi_{38} = .1009429577, && \varphi_{39} = .1017233388, \\
&\varphi_{40} = .1033313491, && \varphi_{41} = .1058685767, && \varphi_{42} = .1095068588, \\
&\varphi_{43} = .1145167936, && \varphi_{44} = .1213147333, && \varphi_{45} = .1305321589, \\
&\varphi_{46} = .1430803899, && \varphi_{47} = .1599940895, && \varphi_{48} = .1809691691
\end{aligned}$$

Note that we do not have to use the parameters $\varphi_i$ when constructing the gear outline. As we now know $r$ we can use a pattern, symmetry, congruence or other techniques to draw the gear in CAD. The result of this example as well as the result using a greater number of teeth can be seen in fig. 9 below.
Due to the symmetry of $\gamma$ and the fact that $4 \mid n$ the resulting gear outline is symmetric along the $x$ and $y$-axis.




(a) $n = 12$ (b) $n = 24$

Figure 9: Resulting elliptical gear with different number of teeth

**Example 3.4.7** Elliptical gear and square gear that fit together

Given the elliptical gear $E$ constructed in example 3.4.6 a fitting square gear $S$ should be calculated.

We recall that $r = 1.513379781$ for $n_E = 24$. We now choose the side-length $s$ of $S$ according to Theorem 3.2 such that $4r \mid s$. Let $s = 28r$.

Note that it is very easy to calculate fitting gears if the size of at least one of them is not given.

As we now know $s$ and $r$ we can proceed to calculate $n_S = 28$ using (again) the findings of Theorem 3.2:

$$n_S = \frac{s}{r}.$$

The two resulting gears are shown in fig. 10.

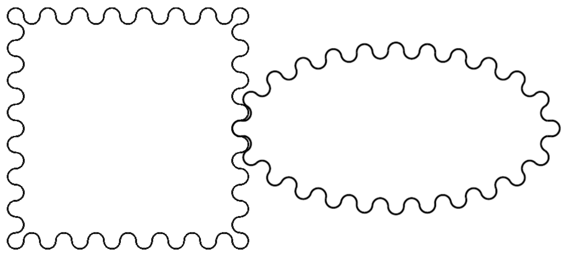


Figure 10: Resulting elliptical gear with fitting square gear

# A. Code used in the Examples

## A.1. Elliptical gear with given size and number of teeth

```
restart;
with(Student[Precalculus]);
d := proc (x, y) options operator, arrow, function_assign; Distance(x
    , y) end proc;
a := 30;
b := 15;

ell := proc (t) options operator, arrow, function_assign; `<,>`(a*cos
    (t), b*sin(t)) end proc;
plot([ell(t)[1], ell(t)[2], t = 0 .. 2*Pi]);

n := 24;
m := 50;
eq := {p[1] = 0};
NULL;
for i to 4*n-1 do unassign('k'); unassign('l'); eq := eq union {d(ell
    (sum(p[k], k = 1 .. i)), ell(sum(p[l], l = 1 .. i+1)))^2 = r^2};
    unassign('k'); unassign('l') end do;
NULL;
eq := eq union {d(ell(sum(p[k], k = 1 .. 4*n)), ell(p[1]))^2 = r^2};
    unassign('k'); unassign('l');
eq;

nb := {p[1] = 0 .. 2*Pi/m};
for i from 2 to 4*n do nb := nb union {p[i] = 0 .. 2*Pi/m} end do;
nb;
vars := {r};
for i to 4*n do vars := vars union {p[i]} end do;
vars;


sol := fsolve(eq, vars, nb);

evalf(sum(sol[k], k = 2 .. 4*n+1));
```

Listing 1: elliptical gears using maple

# References

Weisstein, Eric W. (2020). *Inscribed Angle.* URL: https://mathworld.wolfram.com/InscribedAngle.html (visited on 05/05/2020) (cit. on p. 6).

# List of Figures